Fundamental Mathematics for Robotics
Homework Set #06-2, Dr.T

[1] Answer the following questions using the same matrices given in Problem [3], Homework Set #06-1, which are:

$$\mathrm{A} = \begin{bmatrix} -3 & -2 \\ 7 & 5 \end{bmatrix}, \mathrm{B} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}, \mathrm{C} = [2 \quad 1], \mathrm{D} = [1]$$

(a) Compute A^{-1}
(b) Find $\vec{x}$ such that $\mathrm{A}\vec{x} = \mathrm{B}$
(c) Find $\vec{x}$ such that $\vec{x}^T A = C$
(d) Can you compute $(\mathrm{BC})^{-1}$? How about $(\mathrm{CB})^{-1}$?
(e) (Extra) Compute $(x\mathrm{I} - \mathrm{A})^{-1}$
(f) (Extra) Compute $\mathrm{C}(x\mathrm{I} - \mathrm{A})^{-1}\mathrm{B} + \mathrm{D}$

[2] Find two 2×2 matrices with the following property:
(a) $\mathrm{M}^{\mathrm{T}} = \mathrm{M}, \quad \mathrm{M} \neq \mathrm{I}$
(b) $\mathrm{M}^{\mathrm{T}} = -\mathrm{M}, \quad \mathrm{M} \neq \Theta$
(c) $\mathrm{M}^{\mathrm{T}} = \mathrm{M}^{-1}$

[3] Show that for any square matrix M (2×2),
(a) $\mathrm{Q} \triangleq \frac{M+M^T}{2}$ is a symmetric matrix.
(b) $W \triangleq \frac{M-M^T}{2}$ is asymmetric matrix.
(c) $M = Q + W$
(d) (Extra) What is the implication of (c)?

[4] In robotics, we often use a matrix $R(\theta) \triangleq \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. Answer the following questions on this matrix.
(a) Compute $\vec{y} = R(\theta)\vec{x}$, where $\vec{x}^T = [x_1 \quad x_2]$.
(b) Compute the norm $||\vec{x}||$.
(c) Compute the norm $||\vec{y}||$.
(d) Compute the inner product $\vec{x} \cdot \vec{y}$ (Note: $< \vec{x}, \vec{y} >$ is also used for the inner product)
(e) Show that $\det R(\theta) = 1$ for any θ.
(f) Compute $R(\theta)^{-1}$.
(g) Show that $R(\theta)^{\mathrm{T}} = R(\theta)^{-1}$.
(h) Show that $R(\theta)^{-1} = R(-\theta)$.
(i) (Extra) Find the angle ϕ between the two vectors $\vec{x}$ *and* $\vec{y}$.
(j) (Extra) Show that the 1st column and the 2nd column of this matrix $R(\theta)$ are orthogonal (normal) regardless of the value of θ. (Hint: What happens to the inner product of two orthogonal vectors?)

(Note: This matrix is called the rotation matrix.)

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Extra Homework Set #06-Extra1, Dr.T

[1] Compute the norm of the following vector:
(a) $\vec{x} = [3 \quad -2]^T$
(b) $\vec{y} = [1 \quad -3]^T$
(c) $\vec{z} = [4 \quad -2 \quad 3]^T$
(d) $\vec{w} = [2 \quad -2 \quad -3]^T$
(e) $\vec{v} = [-1 \quad 4 \quad 2 \quad -3]^T$
(f) $\vec{u} = [3 \quad -1 \quad 4 \quad -2]^T$

[2] Compute the inner products of the vectors in Problem [1] for all possible combinations.

[3] Compute the matrix products of pairs of matrices among the following matrices when possible.

$$\mathrm{L} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix},\ \mathrm{M} = \begin{bmatrix} 2 & -2 & 1 \\ 3 & 1 & -2 \end{bmatrix},\ \mathrm{N} = \begin{bmatrix} -2 & 1 \\ -1 & -2 \\ 1 & 1 \end{bmatrix} \text{ and } \mathrm{P} = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

[4] Compute the matrix-vector products of the vectors in Problem [1] and the matrices in Problem [3] for all possible combinations.

[5] Compute the determinants of the matrices in Problem [3] if possible.

[6] Compute the inverse matrices of the matrices in Problem [3] if possible.

[7] (Extra) Now, let us consider powers of a matrix, namely, M^n for a matrix M. Discuss how we should define it and what the condition(s) should be.

[8] (Extra) If $x^2 = 0$ for a scalar x, then $x = 0$. It is not true for a matrix M, i.e., even if $M^2=\Theta$ M may not be Θ. Here Θ is a zero matrix whose elements are all zero. Give an example of non-zero 2x2 matrix M for which $M^2=\Theta$.

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Extra Homework Set #06-Extra2, Dr.T

[1] Compute the norm of the given vectors. Also, compute inner products of all possible combinations.

$$\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \quad \vec{z} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \text{ and } \vec{u} = \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix}$$

[2] Compute the inverse matrix if you can.

(a) $A_1 = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$, (b) $A_2 = \begin{bmatrix} -4 & 1 \\ -1 & -2 \end{bmatrix}$, (c) $A_3 = \begin{bmatrix} 3 & 2 \\ -6 & -4 \end{bmatrix}$, (d) $A_4 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

(e) (Extra) $A_5 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -3 \end{bmatrix}$, (f) (Extra) $A_6 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$

[3] Given a 2x2 matrix A, $A = \begin{bmatrix} -3 & -4 \\ 5 & 6 \end{bmatrix}$, answer the following questions:

(a) Compute A^2
(b) Compute $3A - 2I$
(c) Compute A^{-1}
(d) Compute $-\frac{1}{2}A + \frac{3}{2}I$
(e) (Extra) Compute A^{-2} using the result of (a), namely, $A^{-2} = (A^2)^{-1}$
(f) (Extra) Compute A^{-2} using the result of (c), namely, $A^{-2} = (A^{-1})^2$
(g) Compute $-\frac{3}{4}A + \frac{7}{4}I$
(h) (Extra) Can you find α and β such that $A^3 = \alpha A + \beta I$