Fundamental Mathematics for Robotics
Homework Set #15, Dr.T

Note: The problems in this HW set are the ones to point the direction we go from this course on. As such, you need a little ingenuity to solve them. Please wonder why these problems are given at the end of this course. You will find the answers as you advance in robotics.

[1] Solve the following problems:

(a) Show that if $x(t) = Ae^{-2t}$, then $\frac{dx(t)}{dt} + 2x(t) = 0$, where A is a constant.

(b) Show that if $x(t) = A\sin 3t$, then $\frac{d^2x(t)}{dt^2} + 9x(t) = 0$, where A is a constant.

(c) Find the value K if $x(t) = Ke^{-3t}$ and its derivatives satisfies the following equation: $\frac{d^2x(t)}{dt^2} + 4\frac{dx(t)}{dt} + 9x(t) = 24e^{-3t}$.

[2] A linear combination of a function and its derivatives is called an **annihilator** of a function if the linear combination becomes 0. An example is the one in Part (a) of the above problem, where $\frac{dx(t)}{dt} + 2x(t)$ is an annihilator for $x(t) = Ae^{-2t}$. Find an annihilator for the following functions.

(a) $x(t) = Be^{3t}$

(b) $x(t) = C\cos 2t$

(c) $x(t) = At + B$

(d) $x(t) = Ae^{-2t} + Be^{3t}$ (Hint: You need to annihilate 2 functions not 1)

[3] There exist non-zero vectors $\vec{x} \neq \vec{0}$ for any square matrix A such that the multiplication of A to $\vec{x}$, $A\vec{x}$, becomes a scalar multiple of $\vec{x}$, $k\vec{x}$, in other words, $A\vec{x} = k\vec{x}$.

(a) Show that $\vec{y} = \alpha\vec{x}$ also satisfies $A\vec{y} = k\vec{y}$ if $A\vec{x} = k\vec{x}$. (This means that the norm of $\vec{x}$ is not the matter, but the direction of $\vec{x}$ is.)

(b) Let us find a vector $\vec{x}$ and a scaler k for $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$. First, let $\vec{x} = [a \;\; b]^T$ and substitute it to the equation $A\vec{x} = k\vec{x}$. Then find k assuming that $a \neq 0 \neq b$. (Hint: there are 2 values for k.)

(c) Find the relationship between a and b for each k value in the form of $b = \cdots$.

(d) Substitute the expression you found in (c) to $\vec{x}$ and express $\vec{x}$ in the form $\vec{x} = a\vec{z}$ for each k value.

(e) Check if the vectors you found in (d) actually satisfy the equation $A\vec{x} = k\vec{x}$. Here, we can assume $a = 1$, WLOG.

(f) (Extra) It is a fact that both the scalar k and the vector $\vec{x}$ can be complex. This is because the real number system is not complete, but the complex number system is. See that this is the case using the following matrix.

$$A = \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}$$