Homework Sets

Associated with

The Video Lecture Series
on
Fundamental Mathematics for Robotics

prepared by

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Fundamental Mathematics for Robotics
Homework Set #01, Dr.T

Problem 1:　Find the Euler's formula and state what it means.　You can use any reference, but you need to quote the reference in your report.

Problem 2:　Cite one of your favorite robots and explain it (including what, when, who, for what, etc.) and tell me why you like that robot.

Fundamental Mathematics for Robotics
Homework Set #02, Dr.T

[1] Find the location of the end-effector (EF) (x_2, y_2) of the two-link manipulator.
  (a) Using the definitions of the coordinate frames and angles shown in Fig.1.
  (b) Similarly using Fig.2.
  (c) (Extra) Similarly using Fig.3. (Note that this is a three-link manipulator.)

[2] Answer the following questions on the given graph in Fig.4 (you can use your own graph paper).
  (a) Construct a first-order polynomial function that passes through both points A and B.
  (b) Construct a second-order polynomial function that passes through both points A and B.
  (c) Construct a third-order polynomial function that passes through both points A and B.
  (d) Construct two functions different from those in (a) through (c) that passes through both points A and B.

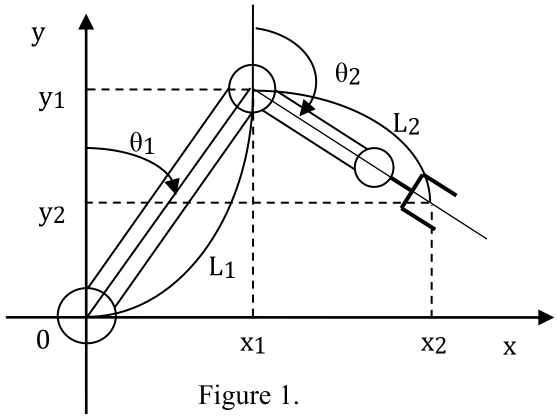


Figure 1.

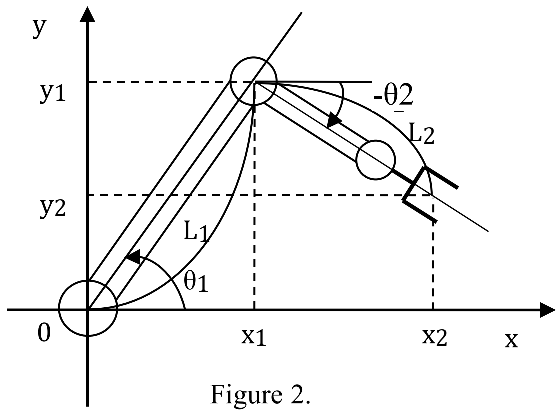


Figure 2.

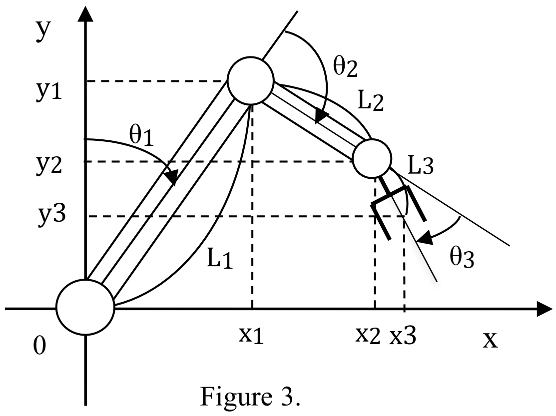


Figure 3.

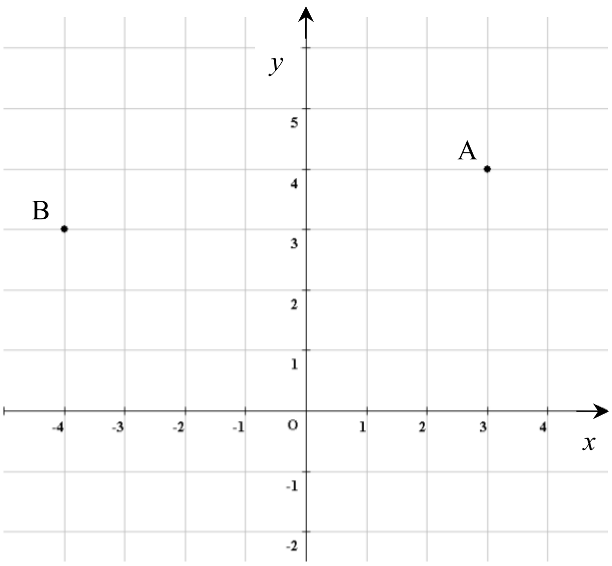
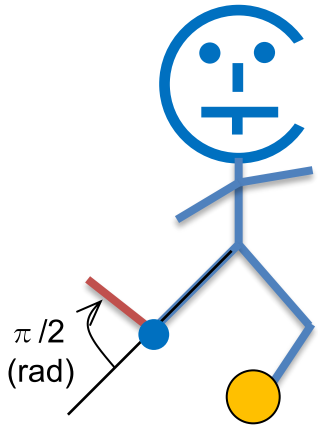


Figure 4.

Fundamental Mathematics for Robotics
Homework Set #03, Dr.T

[1] Suppose that the motion of the knee joint for Robo-Kick 1 is needed. Here the initial and the final conditions are $\theta(0) = \pi/2$ (rad), $\dot{\theta}(0) = 0, \theta(3) = 0(\text{rad}), \dot{\theta}(3) = 0$. Design a trajectory following what we learned in the lecture. (Several notes are due here. The direction of the knee joint is taken clockwise because the range of motion of the knee joint is lopsided and it make sense to make the direction of motion to be positive. It is also noted that rad (radian) is used more often in robotics.)



[2] Let us try a different set of assumptions for Robo-Kick 2.

1. The direction of measuring the hip joint is reversed.
2. Let the initial condition be $\theta(0) = \pi/3$ (rad), $\dot{\theta}(0) = 0$.
3. Let the final condition be $\theta(3) = -\pi/6$ (rad), $\dot{\theta}(3) = 0$.

Find a trajectory in the form of a 3rd order polynomial and draw the trajectory and the speed.

[3] Let us try a different function for Robo-Kick 2. Since the graph of the resulting 3rd order polynomial looks like a half of one period of cosine function, how about trying $\theta(t) = A + B\cos Ct$, where $A, B,$ and C are parameters.

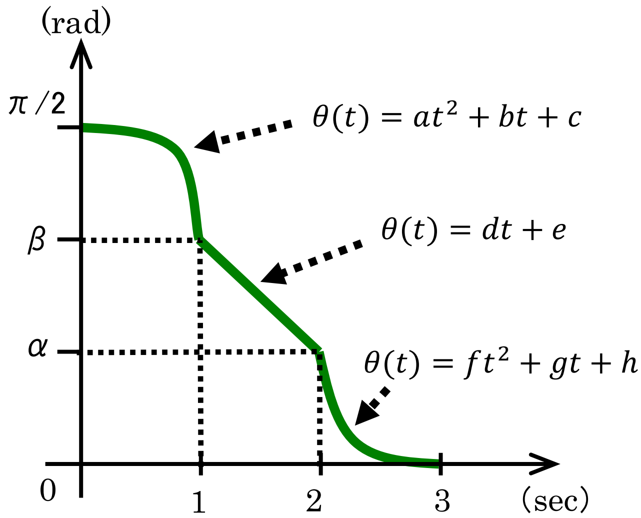
(a) Find the parameter values using four constraints given in the class, namely $\theta(0) = -60(\text{deg}), \dot{\theta}(0) = 0, \theta(3) = 30(\text{deg}), \dot{\theta}(3) = 0$.
(b) Plot or sketch the trajectory you found in (a).
(c) Compute and plot (or sketch) the speed of the trajectory.
(d) (Extra) Compute and plot (or sketch) the acceleration of the trajectory.
(e) Does the number of parameters match the number of constraints? Obviously not. Explain why we can find an answer? (A hint is in the problem statement.)

[4] Let us find the units of the quantities appearing the following equation, where $f(t)$ is in kg・m/sec² and $\theta(t)$ is in rad:

$$f(t) = M\ddot{\theta}(t) + D\dot{\theta}(t) + K\theta(t)$$

Fundamental Mathematics for Robotics
Homework Set #04, Dr.T

[1] Let us repeat the Robo-Kick problem with different set-up. Suppose that the given figure is a graph of the angle $\theta(t)$ of the knee joint, where **a**, **b**, **c**, **d**, **e**, **f**, **g**, and **h** are parameters and α and β are adjustable angles.



(a) The graph of $\theta(t)$ shows both the initial and final conditions, $\theta(0) = \pi/2$ (rad) and $\theta(3) = 0$ (rad). Use them to find two equations.

(b) The joint angle must be continuous at time $t = 1$ and $t = 2$ (sec). Use $t-$ and $t+$ notations to express those constraints and write two equations.

(c) The graph also shows that the values of the joint angle $\theta(t)$ is β (rad) at $t = 1$ (sec) and α (rad) at $t = 2$ (sec). Use this information to write two equations.

(d) The joint was not moving before time $t = 0$ (sec) and this implies that the angular speed at $t = 0$ (sec) is zero. Use this constraint to write an equation.

(e) It seems that the slope of the graph at $t = 3$ (sec) is zero, namely, the knee joint stops. Use this constraint to write an equation.

(f) Find **a**, **b**, **c**, **d**, **e**, **f**, **g** and **h** from the above conditions using adjustable angles α and β.

(g) Choose several (at least three) pairs of α and β and plot (using a PC) or sketch (handwriting) the angle $\theta(t)$ for those values of α and β where

$$0 < \alpha < \beta < \pi/2$$

(h) (Extra) Suppose we want to make the trajectory 'smooth,' meaning that the speed is also continuous, i.e., the derivative $d\theta(t)/dt$ is continuous all the time. Write two equations of continuity conditions for the derivative at times $t = 1$ and 2 (sec). Find the values of α and β that realizes the continuity.

(i) (Extra) what should be the units of parameters **a** through **h**? Hint: The unit of t is (sec) and that of $\theta(t)$ is (rad).

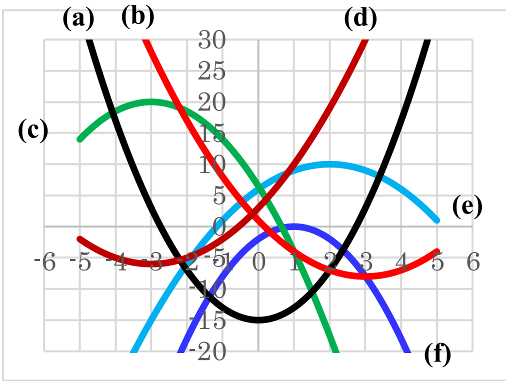
[2] Let's design original trajectories for the knee joint of a robot. We keep the total time interval of 3 (sec), the initial angle of $\pi/2$ (rad), the final angle of 0 (rad), and the initial speed of 0 (rad/sec). You can change

(1) the number of subintervals,
(2) the lengths of subintervals, and
(3) the types of functions used in subintervals.

Make sure you plot or sketch your trajectories. Hint: In Problem [1], we used 3 subintervals, equal subinterval length of 1 (sec), and polynomials, respectively.

Fundamental Mathematics for Robotics
Homework Set #05-1, Dr.T

[1] The given figure shows graphs of $f(x) = Ax^2 + Bx + C$. Find $A, B,$ and C of each graph.



[2] Plot or sketch the functions $\sin\theta$ and $\cos\theta$ for $-2\pi \leq \theta \leq 2\pi$ on a single graph.

[3] Plot or sketch the following functions:
(a) $\cos 2\theta$ for $-\pi \leq \theta \leq \pi$
(b) $-\cos 2\theta$ for $-\pi \leq \theta \leq \pi$
(c) $1 - \cos 2\theta$ for $-\pi \leq \theta \leq \pi$
(d) $\frac{1}{2}(1 - \cos 2\theta)$ for $-\pi \leq \theta \leq \pi$
(e) Discuss what the resulting graph is.

[4] Plot or sketch the following functions:
(a) $f(x) = \frac{x^2+3}{x^2+1}$ on $[-10, 10]$
(b) $g(x) = \begin{cases} -x^2 + 3, & x < 1 \\ x^2 - 4x + 5, & 1 < x \end{cases}$ on $[-1, 3]$

[5] Plot or sketch the exponential function e^{at+b} for $a = -1, 0, 1$ *and* $b = -1, 0, 1$ over the interval [-2, 2] on a single graph. (How many curves do you need to plot or sketch?)

[6] Find two functions $f(x)$ that satisfy all of the following conditions. One of them should contain exponential functions only (you can use all four arithmetic operations with constants).
i. $f(x)$ is continuous.
ii. $f(x)$ goes to 0 as x goes to ∞.
iii. $f(x)$ goes to 1 as x goes to $-\infty$.
iv. $f(x)$ is monotone decreasing.

[7] Use the Euler's formula to prove the double angle formula given by

$$\cos(2x) = \cos^2(x) - \sin^2(x)$$

[8] (Extra) Use the Euler's formula to prove the following formula:

$$\sin(x) - \sin(y) = 2\cos(\frac{x+y}{2})\sin(\frac{x-y}{2})$$

Fundamental Mathematics for Robotics
Homework Set #05-2, Dr.T

[1] Let us consider numerical examples of a two-link manipulator given in the lecture. Let us use the following values: $L_1 = 50(cm), L_2 = 30(cm)$. Compute the end-effector (EF) position (x_2, y_2) for the following joint angles.

(a) $\theta_1 = 30(deg), \theta_2 = 60(deg)$
(b) $\theta_1 = 60(deg), \theta_2 = 30(deg)$
(c) $\theta_1 = -30(deg), \theta_2 = 60(deg)$
(d) $\theta_1 = 30(deg), \theta_2 = -60(deg)$
(e) $\theta_1 = -13.57(deg), \theta_2 = 60(deg)$
(f) What is the relationship between (d) and (e)?

[2] Let us fill in derivations that were skipped in the lecture. From the 2-Link equation:

$x_2 = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) \quad (1)$

$y_2 = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) \quad (2)$

(a) Find $A_1, A_2, B_1, and\ B_2$ in $(1) \rightarrow\ x_2 = A_1 \sin\theta_1 + A_2 \cos\theta_1 \quad (3)$, $(2) \rightarrow\ y_2 = B_1 \sin\theta_1 + B_2 \cos\theta_1 \quad (4)$

(b) Derive $C_1 \sin\theta_1 + C_2 \cos\theta_1 = 0$ by eliminating l.h.s. from (3) and (4).

(c) Show that $\theta_1 = \tan^{-1}\frac{-C_2}{C_1}$.

[3] Let us do the Inverse-Kinematics computation for the 2-Link manipulator in Problem [1]. Can you compute θ_1, and θ_2 from the following EF positions (x_2, y_2).

(a) $x_2 = 73.3, y_2 = 25$
(b) $x_2 = 58.3, y_2 = 50.98$
(c) $x_2 = 10.0, y_2 = 69.28$
(d) $x_2 = 65.36, y_2 = 35.36$
(e) $x_2 = 60.0, y_2 = 60.0$

[4] Solve the following questions on trigonometric functions using inverse trigonometric functions. Use the region $-\pi < \theta \leq \pi$. You must specify the quadrant in which your answer is located.

(a) Solve for the angle θ: $5\cos\theta + 3 = 0$
(b) Solve for the angle θ: $5\sin\theta + 4 = 0$
(c) Use ATAN2(x,y) to solve for the angle θ: $4\cos\theta + 3\sin\theta = 0$ (Hint: Which quadrant?)
(d) Prove: $\cos^2\theta = 1/(1 + \tan^2\theta)$
(e) (Extra) Use the result of (d) for (a) to use ATAN2 to solve for the angle θ.

Fundamental Mathematics for Robotics
Homework Set #06-1, Dr.T

[1] Given a vector matrix equation: $M\vec{v} = \vec{u}$ with the following two vectors and a matrix. $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\vec{u} = \begin{bmatrix} e \\ k \end{bmatrix}$ *and* $M = \begin{bmatrix} a & b & c \\ f & g & h \end{bmatrix}$, where $a, b, c, e, f, g\ h,$ and k are scalars.

(a) Write the corresponding set of linear simultaneous equations.

(b) (Extra) Can you solve this set of equations? (Remember, you need to explain!)

[2] Given the following set of simultaneous equations.

$$ax + by = c$$
$$ex + fy = g$$
$$hx + ky = m$$

(a) Write a vector matrix equation of the form $\mathrm{H}\vec{\mathrm{p}} = \vec{\mathrm{q}}$ corresponding to this set of simultaneous equations, i.e., define two vectors $\vec{\mathrm{p}}$ and $\vec{\mathrm{q}}$ and a matrix H.

[3] Given the following matrices:

$$\mathrm{A} = \begin{bmatrix} -3 & -2 \\ 7 & 5 \end{bmatrix}, \mathrm{B} = \begin{bmatrix} -1 \\ 7 \end{bmatrix}, \mathrm{C} = [2 \quad 1], \mathrm{D} = [1]$$

Compute the following quantities if possible.

(a) $2\mathrm{B} + \mathrm{C}^\mathrm{T}$ (b) AB (c) $\mathrm{A}^2\mathrm{B}$ (d) $\mathrm{AB} + \mathrm{C}^\mathrm{T}$ (e) CA (f) $\mathrm{B}^\mathrm{T} - \mathrm{CA}$ (g) BC
(h) $-2\mathrm{A} + \mathrm{BC}$ (i) CB (j) $\mathrm{CB} - \mathrm{D}$ (k) A(BC) (l) (AB)C (m) xA (n) $xI - A$
(o) (Extra) Make up 2 combinations of those matrices that are different from the above and compute them.

[4] (Extra) Show that $(MN)^T = N^T M^T$ using 2x2 matrices $\mathrm{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\mathrm{N} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$.

Fundamental Mathematics for Robotics
Homework Set #06-2, Dr.T

[1] Answer the following questions using the same matrices given in Problem [3], Homework Set #06-1, which are:

$$A = \begin{bmatrix} -3 & -2 \\ 7 & 5 \end{bmatrix}, B = \begin{bmatrix} -1 \\ 7 \end{bmatrix}, C = \begin{bmatrix} 2 & 1 \end{bmatrix}, D = [1]$$

- (a) Compute A^{-1}
- (b) Find $\vec{x}$ such that $A\vec{x} = B$
- (c) Find $\vec{x}$ such that $\vec{x}^T A = C$
- (d) Can you compute $(BC)^{-1}$? How about $(CB)^{-1}$?
- (e) (Extra) Compute $(xI - A)^{-1}$
- (f) (Extra) Compute $C(xI - A)^{-1}B + D$

[2] Find two 2×2 matrices with the following property:

- (a) $M^T = M, \quad M \neq I$
- (b) $M^T = -M, \quad M \neq \Theta$
- (c) $M^T = M^{-1}$

[3] Show that for any square matrix M (2×2),

- (a) $Q \triangleq \frac{M+M^T}{2}$ is a symmetric matrix.
- (b) $W \triangleq \frac{M-M^T}{2}$ is asymmetric matrix.
- (c) $M = Q + W$
- (d) (Extra) What is the implication of (c)?

[4] In robotics, we often use a matrix $R(\theta) \triangleq \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. Answer the following questions on this matrix.

- (a) Compute $\vec{y} = R(\theta)\vec{x}$, where $\vec{x}^T = [x_1 \quad x_2]$.
- (b) Compute the norm $\|\vec{x}\|$.
- (c) Compute the norm $\|\vec{y}\|$.
- (d) Compute the inner product $\vec{x} \cdot \vec{y}$ (Note: $< \vec{x}, \vec{y} >$ is also used for the inner product)
- (e) Show that $\det R(\theta) = 1$ for any θ.
- (f) Compute $R(\theta)^{-1}$.
- (g) Show that $R(\theta)^T = R(\theta)^{-1}$.
- (h) Show that $R(\theta)^{-1} = R(-\theta)$.
- (i) (Extra) Find the angle ϕ between the two vectors $\vec{x}$ and $\vec{y}$.
- (j) (Extra) Show that the 1st column and the 2nd column of this matrix $R(\theta)$ are orthogonal (normal) regardless of the value of θ. (Hint: What happens to the inner product of two orthogonal vectors?)

(Note: This matrix is called the rotation matrix.)

Fundamental Mathematics for Robotics
Extra Homework Set #06-Extra1, Dr.T

[1] Compute the norm of the following vector:

(a) $\vec{x} = [3 \quad -2]^T$
(b) $\vec{y} = [1 \quad -3]^T$
(c) $\vec{z} = [4 \quad -2 \quad 3]^T$
(d) $\vec{w} = [2 \quad -2 \quad -3]^T$
(e) $\vec{v} = [-1 \quad 4 \quad 2 \quad -3]^T$
(f) $\vec{u} = [3 \quad -1 \quad 4 \quad -2]^T$

[2] Compute the inner products of the vectors in Problem [1] for all possible combinations.

[3] Compute the matrix products of pairs of matrices among the following matrices when possible.

$$\mathrm{L} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix},\ \mathrm{M} = \begin{bmatrix} 2 & -2 & 1 \\ 3 & 1 & -2 \end{bmatrix},\ \mathrm{N} = \begin{bmatrix} -2 & 1 \\ -1 & -2 \\ 1 & 1 \end{bmatrix} \text{ and } \mathrm{P} = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

[4] Compute the matrix-vector products of the vectors in Problem [1] and the matrices in Problem [3] for all possible combinations.

[5] Compute the determinants of the matrices in Problem [3] if possible.

[6] Compute the inverse matrices of the matrices in Problem [3] if possible.

[7] (Extra) Now, let us consider powers of a matrix, namely, M^n for a matrix M. Discuss how we should define it and what the condition(s) should be.

[8] (Extra) If $x^2 = 0$ for a scalar x, then $x = 0$. It is not true for a matrix M, i.e., even if $M^2=\Theta$ M may not be Θ. Here Θ is a zero matrix whose elements are all zero. Give an example of non-zero 2x2 matrix M for which $M^2=\Theta$.

Fundamental Mathematics for Robotics
Extra Homework Set #06-Extra2, Dr.T

[1] Compute the norm of the given vectors. Also, compute inner products of all possible combinations.

$$\vec{x}=\begin{bmatrix}2\\3\end{bmatrix},\quad \vec{y}=\begin{bmatrix}3\\-2\end{bmatrix},\quad \vec{z}=\begin{bmatrix}-2\\4\end{bmatrix},\quad \vec{w}=\begin{bmatrix}1\\2\\3\end{bmatrix},\quad \vec{v}=\begin{bmatrix}3\\-2\\1\end{bmatrix}\ \text{and}\ \vec{u}=\begin{bmatrix}-7\\2\\1\end{bmatrix}$$

[2] Compute the inverse matrix if you can.

(a) $A_1=\begin{bmatrix}1 & 3\\4 & 2\end{bmatrix}$, (b) $A_2=\begin{bmatrix}-4 & 1\\-1 & -2\end{bmatrix}$, (c) $A_3=\begin{bmatrix}3 & 2\\-6 & -4\end{bmatrix}$, (d) $A_4=\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}$

(e) (Extra) $A_5=\begin{bmatrix}2 & 0 & 0\\0 & 4 & 0\\0 & 0 & -3\end{bmatrix}$, (f) (Extra) $A_6=\begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\-6 & -11 & -6\end{bmatrix}$

[3] Given a 2x2 matrix A, $A=\begin{bmatrix}-3 & -4\\5 & 6\end{bmatrix}$, answer the following questions:

(a) Compute A^2
(b) Compute $3A-2I$
(c) Compute A^{-1}
(d) Compute $-\frac{1}{2}A+\frac{3}{2}I$
(e) (Extra) Compute A^{-2} using the result of (a), namely, $A^{-2}=(A^2)^{-1}$
(f) (Extra) Compute A^{-2} using the result of (c), namely, $A^{-2}=(A^{-1})^2$
(g) Compute $-\frac{3}{4}A+\frac{7}{4}I$
(h) (Extra) Can you find α and β such that $A^3=\alpha A+\beta I$

Fundamental Mathematics for Robotics
Homework Set #07-1, Dr.T

[1] Evaluate the average speed V at $t = \tau$ over the time interval T.

(a) $f(t) = 3t - t^2$
(b) $f(t) = t^3 - 2$
(c) $f(t) = \sqrt{2t}$
(d) $f(t) = \frac{3}{t+2}$
(e) $f(t) = \frac{3t-5}{t+3}$
(f) (Extra) $f(t) = \sin t$
(g) (Extra) $f(t) = \cos t$
(h) (Extra) $f(t) = e^t$

[2] Compute and make a table of the average speed of a joint angle $\theta(t)$ given by $\theta(t) = (t-1)^2$ at a time τ and over an interval T with the following specifications:

(a) At $\tau = 1$ with $T = 1, 0.5, 0.1, 0.05, 0.01$ and 0.001
(b) At $\tau = 0.5$ to 1.5 at 0.1 interval with $T = 0.1$

[3] Evaluate the expression $\Delta \triangleq \frac{f(x) - f(x-h)}{h}$ for the following functions $f(\cdot)$. Can you cancel h from the numerator and the denominator of the fraction?

(a) $f(x) = -2x + x^2$
(b) $f(x) = \sqrt{3x}$
(c) $f(x) = \frac{2x-3}{x+4}$
(d) (Extra) $f(x) = x^3 - 2x$

[4] Find the instantaneous speed v at $t = t_0$ using the limit operation. (Hint: In Problem [3], you already have computed the expression $\Delta \triangleq \frac{f(x) - f(x-h)}{h}$ of some of the following functions and cancelled h from the numerator and the denominator.)

(a) $f(t) = -2t + t^2$
(b) $f(t) = \sqrt{3t}$
(c) $f(t) = \sqrt{3\sqrt{t}}$
(d) $f(t) = \frac{2t-3}{t+4}$
(e) (Extra) $f(t) = \frac{3}{(t+2)^2}$

Fundamental Mathematics for Robotics
Homework Set #07-2, Dr.T

Note: Turn in your solutions by the start of the Recitation class on June 9, 2015.

[1] Evaluate the derivative of the following functions using the shortcut Laws. Remember that you can use up to two Laws in a single step and you must state which Laws are used.

(a) $f(x) = (x-2)^2$
(b) $x(t) = \cos(3t-4)$
(c) $y(t) = 3\cos(4t+3) + 2\ \sin(-2t+1)$
(d) $z(t) = 3\ \sin(-t^2+3t-2)$
(e) $g(y) = e^{-y+3} + y^2$
(f) $h(z) = ze^{2z+3}$
(g) (Extra) $g(t) = 2\cos\{\theta(t)\}$
(h) (Extra) $h(x) = e^{-e^{2x+3}}$

[2] Find the derivative of the following function using the definition of the differentiation. Hint: You can use the following equalities:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1, \qquad \lim_{x\to 0}\frac{1-e^{-x}}{x} = 1$$

(a) $f(\theta) = \sin 2\theta$
(b) $f(t) = e^{-t}$
(c) (Extra) $f(\theta) = \cos\theta$

[3] The two definitions of average speed at time t over the interval Δt shown below are, in fact, equivalent when computing the instantaneous speed. Confirm this by computing the instantaneous angular speed $d\theta/dt$ for $\theta(t) = (t+1)^3$ using both definitions.

$$\frac{\theta(t+\Delta t)-\theta(t)}{\Delta t} \text{ and } \frac{\theta(t)-\theta(t-\Delta t)}{\Delta t}$$

[4] (Extra) (This problem is designed to relax your brain. Enjoy!)

(a) Show that if $r = 1/(r-1)$, then $q = 1/(q+1)$ for $q = 1/r$. Also, find r and q.
(b) Find the number x: $x = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cdots}}}}$ (Continued fraction)

Fundamental Mathematics for Robotics
Extra Homework Set #07-Extra1, Dr.T

[1] Compute the derivative of the following function using the definition of the differentiation.

(a) $f(t) = -2t^2 + 3t - 4$
(b) $g(t) = \frac{3}{t+2}$
(c) $v(t) = \sqrt{3-t}$
(d) $w(t) = (t-2)^3$
(e) $y(t) = 3t\sqrt{t}$
(f) (Extra) $x(t) = \frac{2}{\sqrt{t}+3}$

[2] Evaluate the derivative of the following function using the definition of the differentiation.

(a) $f(t) = -2(2t+3)^2 + 3(2t+3) - 4$
(b) $x(t) = \frac{t}{t+3}$
(c) $v(t) = \sqrt{t^2+3}$
(d) $\psi(t) = \frac{4}{5+t^2}$
(e) $\omega(t) = \frac{2}{\sqrt{t+3}}$
(f) (Extra) $\phi(t) = \frac{t}{\sqrt{1-4t^2}}$

[3] Evaluate the derivative of the following function using the shortcut Laws. Remember that you can use up to two Laws in one step.

(a) $\theta(t) = \sin 3t \cdot \cos 4t$
(b) $x(t) = \mathrm{e}^{-4t}\cos(3t)$
(c) $\theta(t) = 3t\sin(-2t+3)$
(d) $y(t) = t^2\cos(-2t^2+3t)$
(e) $\theta(t) = t^2\mathrm{e}^{4t}$
(f) $\varphi(t) = t^2\mathrm{e}^{4t^2}$
(g) $v(t) = 2t^2\sqrt{2-t}$
(h) $\phi(t) = \frac{2t+6}{(t-2)^2}$
(i) $w(t) = \frac{t}{\sqrt{1-4t^2}}$
(j) $\psi(t) = \frac{4}{3+\cos(3t)}$
(k) (Extra) $z(t) = 5t\mathrm{e}^{4t}\cos(3t-2)$

Fundamental Mathematics for Robotics
Extra Homework Set #07-Extra2, Dr.T

[1] Find the indicated derivative of the given function.

(a) $f(x) = 3x^4 - 2x^3 + x^2 - 2x + 4, \quad f^{(3)}(x)$

(b) $y(x) = \frac{1}{\sqrt{x}}, \quad y^{(4)}(x)$

(c) $g(t) = \sin 3t + \cos 2t, \quad g^{(2)}(t)$

(d) $\theta(t) = \cos 2t \sin 3t, \quad \ddot{\theta}(t)$

(e) (Extra) $h(t) = e^{-t}\sin 2t, \quad \ddot{h}(t)$

[2] Let us consider differentiating a quotient $f(x)/g(x)$.

(a) Differentiate $h(x) = 1/g(x)$ using the chain rule.

(b) Differentiate $p(x) = f(x)h(x)$ using Law (d).

(c) Obtain Law (h) for Differentiation of Quotient using the results in part (a) and (b).

[3] Let us consider exponential functions with a general base α.

(a) Show that $\alpha^x = e^{\{ln(\alpha)\}x}$. Hint: $y = e^x \iff x = ln\, y$.

(b) Use (a) to compute the derivative of $f(x) = \alpha^x$.

(c) Plot $f(x) = \alpha^x$ for various values of α, α = 1, 1.5, 2, 2.5, e, 3, 5, 10 over the interval [-2, 2]. Hint: Use [1, 10] for the range of y-axis.

(d) Plot $f(x)$ and its derivative pairwise for the α values given in Part (c) over the interval [-2, 2]. Discuss how the relationship between the function and the derivative changes.

[4] (Extra) (This problem is designed to relax your brain. Enjoy!)

(a) Can you compute the number x given by $x = \sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdots}}}$, an infinitely repeated power? (Hint1: Look at the pattern! Hint2: What should be the answer to the question starting with 'Can you ...?')

Fundamental Mathematics for Robotics
Homework Set #08-1, Dr.T

[1] Recall that we have computed the end-effector position of the two-link manipulator in Fig. 1 in Recitation Set #02.

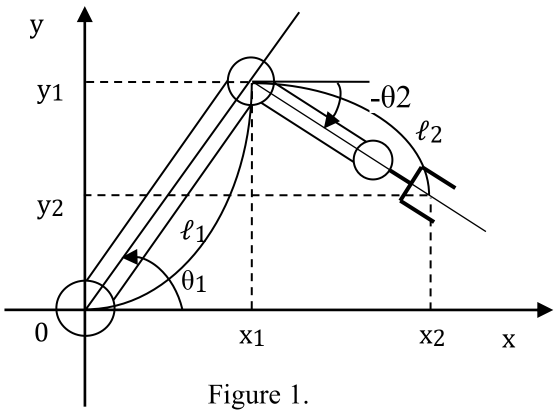


Figure 1.

(a) Compute the partial derivatives $\partial x_1/\partial\theta_1$ and $\partial y_1/\partial\theta_1$.
(b) Compute the partial derivatives $\partial x_1/\partial\theta_2$ and $\partial y_1/\partial\theta_2$.
(c) Compute the partial derivatives $\partial x_2/\partial\theta_1$ and $\partial y_2/\partial\theta_1$.
(d) Compute the partial derivatives $\partial x_2/\partial\theta_2$ and $\partial y_2/\partial\theta_2$.
(e) Compute the x and y directional speeds dx_2/dt and dy_2/dt of the end-effector.

[2] (Extra) Can you find the acceleration of the end-effector of the manipulator in Problem [1]?

[3] Compute the indicated partial derivatives of the given function:

$$g(x, A, \alpha, \beta, \omega) = Ae^{\alpha x}\cos(\omega x + \beta),$$

where the variables $x, A, \alpha, \beta,$ and ω are independent.

(a) $\partial g/\partial x$ (b) $\partial g/\partial A$ (c) $\partial g/\partial\alpha$ (d) $\partial g/\partial\beta$ (e) $\partial g/\partial\omega$

[4] Repeat [3] for the following function where x, y, z are variables but $A, \alpha, \beta,$ and ω are constant parameters this time:

$$h(x, y, z) = Ae^{\alpha\sqrt{x^2+y^2}}cos(\omega z + \beta)$$

Fundamental Mathematics for Robotics
Homework Set #08-2, Dr.T

[1] Compute the Jacobian matrix of the following vector functions.

(a) $\begin{bmatrix} x(\theta_1,\theta_2) \\ y(\theta_1,\theta_2) \end{bmatrix} = \begin{bmatrix} 2cos\ (\theta_1) + 4cos(\theta_1+\theta_2) \\ 3sin\ (\theta_1) + 2sin(\theta_1+\theta_2) \end{bmatrix}$

(b) $\begin{bmatrix} x(\theta_1,\theta_2,\theta_3) \\ y(\theta_1,\theta_2,\theta_3) \end{bmatrix} = \begin{bmatrix} 2cos\ (\theta_1) + 4cos(\theta_1+\theta_2) + 3cos(\theta_1+\theta_2+\theta_3) \\ 3sin\ (\theta_1) + 2sin(\theta_1+\theta_2) + 4sin(\theta_1+\theta_2+\theta_3) \end{bmatrix}$

(c) (Extra) $\begin{bmatrix} x(\theta_1,\theta_2) \\ y(\theta_1,\theta_2) \\ z(\theta_1,\theta_2) \end{bmatrix} = \begin{bmatrix} -3cos\ (\theta_1+\theta_2) + 2cos(\theta_1-\theta_2) \\ 2cos\ (\theta_1+\theta_2) - 3sin(\theta_1-\theta_2) \\ -\ sin(\theta_1+\theta_2) + 4\ sin(\theta_1-\theta_2) \end{bmatrix}$

[2] Compute the total differential for the following function:

(a) $f(x,y) = tan(4x^2 - 3xy + 6y^2)$

(b) $h(x,y,z) = Ae^{\alpha x^2} cos(\omega z + \gamma) + \mathrm{B}e^{\beta y^2} sin(\omega z + \gamma)$

(c) (Extra) $\begin{bmatrix} x(\theta_1,\theta_2) \\ y(\theta_1,\theta_2) \end{bmatrix} = \begin{bmatrix} 3\ sin(\theta_1) - 2\ sin(\theta_1+\theta_2) \\ -4\ cos(\theta_1) + 5\ cos(\theta_1+\theta_2) \end{bmatrix}$ Hint: The total differential is a vector.

Fundamental Mathematics for Robotics
Homework Set #09, Dr.T

[1] Suppose that an angular speed of a joint is given by the following equation:

$$\omega(t) = 3 - 2\text{t}, \quad 0 \le t$$

Also suppose that the joint is at the origin, i.e., $\theta(0) = 0$ until time $t = 0$.

(a) Write $\theta(2)$ in the form of the Riemann sum with $the\ number\ of\ intervals = N$.
(b) Compute $\theta(2)$ using the Riemann sum at $T = 0.2\ (or\ N = 10)$.
(c) Repeat (b) with $T = 1, 0.1,$ and 0.01.
(d) Compute $\theta(2)$ by taking the limit of $T \to 0$ (or $N \to \infty$) of the Riemann sum.

[2] Repeat Problem [1] with $\omega(t) = t^2 - 2\text{t}, \quad 0 \le t.$

[3] (Extra) How should we modify the solution of Problem [1] if the joint is initially at $\theta(0) = 10$?

[4] Discuss the following issues. The summation in the Riemann sum is up to $N - 1$. Why this is not N?

[5] Prove the following: $\text{S} = \sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

[6] (Extra) Prove the following: $\text{S} = \sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

[7] (Extra) Suppose that the angle $\theta(t)$ and the angular speed $\omega(t)$ were 0 at time $t = 0$ and that the angular acceleration $\dot{\omega}(t)$ is given by the following:

$$\dot{\omega}(t) = 3, \quad 0 \le t \le 2$$

(d) Compute the angular speed at time 2, i.e., $\omega(2)$ using the time interval $T = 0.5$.
(e) Compute the angle at time 2, i.e., $\theta(2)$ using the time interval $T = 0.5$.
(f) Repeat parts (a) and (b) using the time interval T=0.05.

Fundamental Mathematics for Robotics
Homework Set #10, Dr.T

[1] Find the following anti-derivatives using the Rules of integration. Make sure that you state the Rules you used.

(a) $\int x^6 dx$
(b) $\int \frac{2}{y^3} dy$
(c) $\int \sqrt{x} dx$
(d) $\int 3z^{2.5} dz$

[2] Repeat Problem [1] with the following:

(a) $\int e^{-2t} dt$
(b) $\int 3e^{-3x} dx$
(c) $\int e^{-4y+3} dy$

[3] Repeat Problem [2] with the following:

(a) $\int 3\cos 2t \, dt$
(b) $\int \sin(-2x) \, dx$
(c) $\int (3\sin 2y - 2\cos 3y) \, dy$

[4] It is convenient to have a formula for reducing the power of x in the integral of the product of x to the n-th power and an exponential function. Prove the following formula that can realize this reduction.

$$\int x^n e^{ax} \, dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} \, dx$$

Fundamental Mathematics for Robotics
Homework Set #11-1, Dr.T

[1] Find the following anti-derivatives.

(a) $\int sin3t \cdot cos3t\, dt$
(b) $\int cos^2 u\, du$
(c) $\int sin2x \cdot sin3x\, dx$
(d) (Extra) $\int sin^3\theta\, d\theta$

[2] Evaluate the following integrals.

(a) $\int \sqrt{x^3 + 2}x^2 dx$
(b) $\int sin^2\theta \cos\theta\, d\theta$
(c) $\int 12(e^x + 1)^5 e^x\, dx$
(d) (Extra) $\int x^2\sqrt{x+1}\, dx$

[3] Evaluate the following integrals.

(a) $\int_{-1}^{1}(2 - x^2)\, dx$
(b) $\int_0^2 (2 - x)^2\, dx$
(c) $\int_1^4 \frac{1}{\sqrt{x}} dx$
(d) $\int_{-1}^{1}(e^t + e^{-t})\, dt$
(e) $\int_0^{2\pi} \sin x\, dx$
(f) (Extra) $\int_0^{2\pi} sin^2 x\, dx$

[4] Let us compute the improper definite integral $I(n) = \int_0^\infty t^{n-1}e^{-t}dt$ through the following steps.

(a) Compute $I(1)$.
(b) Compute $I(2)$. (Hint: The last problem of HW Set #10)
(c) Find the relationship between $I(2)$ and $I(1)$.
(d) Find the relationship between $I(n+1)$ and $I(n)$.
(e) Use the result in part (d) recursively to find $I(n+1)$.

Fundamental Mathematics for Robotics
Homework Set #11-2, Dr.T

[1] Find the indefinite integral (anti-derivative) $F(x)$ of the following functions.

(a) $f(x) = \frac{2}{(x+2)(x+3)}$

(b) $f(x) = \frac{3x+7}{(x+2)(x+3)}$

(c) $f(x) = \frac{x^2+8x+13}{(x+2)(x+3)}$

[2] Let us compute the angle of a joint of a robot from the following torque profile. Let us assume that the moment of inertia $I = 1/2$.

$$\tau(t) = \begin{cases} \pi/6, & 0 \le t < 2 \\ 0, & 2 \le t < 5 \\ -\pi/9, & 5 \le t \le 8 \end{cases}$$

Note that the torque =0 outside of the given interval.

(a) Plot the torque profile.

(b) Find the angular speed $\omega(t)$ assuming that the joint is at rest at time $t = 0$.

(c) Plot the angular speed.

(d) Find the joint angle $\theta(t)$ assuming that the initial angle is $-\pi/2$ at time $t = 0$.

(e) Plot the angle $\theta(t)$.

[3] Compute the following integrals:

(a) Definite integral: $F(t) = \int_0^t 3(t-u)du$

(b) Definite integral: $F(t) = \int_0^t 4e^{t-u}du$

(c) Indefinite integral: $I = \int e^{at} \cos bt\, dt$

(d) (Extra) Indefinite integral: $I = \int \frac{1}{x^2+1} dx$

(e) (Extra) Indefinite integral: $I = \int \frac{1}{\cos^2\theta} d\theta$

Fundamental Mathematics for Robotics
Homework Set #12, Dr.T

[1] Find local minimum(s), local maximum(s), and inflection point(s) of the following functions if they exist:

(a) $f(x) = 2x^3 + 3x^2 - 36x + 8$
(b) $x(t) = 5t^2 + (15 - 2t^2)$
(c) $f(x, y) = x^2 + 2xy + 2y^2 - 6x - 4y + 6$
(d) $f(x, y) = 5x^2 + 6xy + 2y^2 - 14x - 8y + 10$
(e) (Extra) $f(x, y) = Ax^2 + 2Bxy + Cy^2 - 2Dx - 2Ey + F$ (all coefficients are positive) Hint: You need to break down into cases.

[2] Repeat Problem [1] with the following functions:

(a) $f(x) = 2(x - 1)/(x^2 + 4)$
(b) $f(x) = (x^2 - 2x + 1)/(x^2 - 2x + 5)$
(c) $f(x) = 3(x - 2)^2/(x^2 + 4)$

[3] This time, let you make your own polynomial functions $p(x)$ that has the following property. Here, the domain is the whole real number, $-\infty$ to ∞.

(a) 1 local minimum and 1 local maximum.
(b) 1 global maximum, 1 local maximum, and 1 local minimum.
(c)

[4] Repeat Problem [3] using functions other than the polynomials.

Fundamental Mathematics for Robotics
Homework Set #13, Dr.T

[1] Show the following Maclaurin expansion.

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \quad \text{for} \quad |x| < 1$$

Here, ignore the region of convergence $|x| < 1$.

[2] Let us show the Maclaurin series expansion of the function $f(x) = 1/(1-x)$ following (a) through (c).

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \quad \text{for} \quad |x| < 1$$

(a) Show $f_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$.

(b) Show the given expansion using $\lim_{n\to\infty} f_n(x) = f(x)$.

(c) Find the condition for the convergence of the series by observing the condition for the convergence of the limit in (b).

[3] (Extra) The convergence of a power series can be checked by the d'Alembert's test for convergence given by the following: A series $\sum_{n=0}^{\infty} a_n$ converges if

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

Use this test to show the range of convergence $|x| < 1$ for x in Problem [1].

[4] Use the d'Alembert's test for convergence to show that the range of convergence for the power series expansion of the exponential function

$$e^{ax} = \sum_{n=0}^{\infty} \frac{a^n}{n!} x^n$$

is $-\infty < x < \infty$.

[5] Find the Maclaurin series expansion of the following functions using the Maclaurin series expansion in Problem [1].

(a) $f(x) = 1/(1+x)$

(b) $f(x) = 1/(1-x^2)$

Fundamental Mathematics for Robotics
Homework Set #14-1, Dr.T

[1] Find a linear regression equation $y = ax + b$ for the following data. You need to construct the extended table. Use three significant digits. Also, compute the RMS error.

x	1.00	2.00	3.00	4.00	5.00	6.00	7.00	8.00	9.00	10.0
y	4.80	10.3	17.2	15.7	23.3	26.2	34.0	31.3	40.5	36.6

[2] Let us use the same data set as Problem [1] but use a different model $x = cy + d$.

(a) Find c^* and d^*.

(b) Find the relationship among $a, b, c,$ and d.

(c) Show that the values of a^* and b^* you found in Problem [1] and the values c^* and d^* do not exactly satisfy the relationship you found in (b).

(d) Discuss why the computed $a^*, b^*, c^*,$ and d^* do not satisfy the relationship exactly.

[3] Repeat Problem [1] for the following data:

x	1.00	2.00	3.00	4.00	5.00	6.00	7.00	8.00	9.00	10.0
y	-25.9	7.94	7.36	37.9	58.2	92.7	146.	197.	254.	329.

Fundamental Mathematics for Robotics
Homework Set #14-2, Dr.T

[1] Find a quadratic regression equation of the form $y = ax^2 + bx + c$ for the data in [3] of HW Set #14-1. Compare and discuss the rms errors here with that in [3].

[2] Consider the following functions that are non-linear in design parameters. Your mission is to linearize those functions in terms of the design parameters so that the linear regression technique can be applied to find the optimum parameter values.

(e) $y = Ae^{bx}$ (Parameters are A and b)
(f) $y = ax^b$ (Parameters are a and b)
(g) $y = 2\cos(\omega x + \phi)$ (Parameters are ω and ϕ)
(h) $y = e^{10^{ax+b}}$ (Parameters are a and b)

[3] The following set of data represents atmospheric absorption of infrared wave. This type of data is used for the evaluation of infrared sensors. It is known that the data follows the double exponential law described by $y(x) = \exp\{10^{ax+b}\}$. Find the optimal set of values a^* and b^* that results in the LMS error.

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
y	0.96	0.94	0.91	0.87	0.81	0.73	0.62	0.48	0.33	0.19	0.08

[4] Find two non-linear models for the following data. Note that the functions are not given this time. You must make up your own functions. Also, you must explain the reason why you chose your functions.

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
y	0.08	0.19	0.33	0.48	0.62	0.73	0.81	0.87	0.91	0.94	0.96

Fundamental Mathematics for Robotics
Homework Set #15, Dr.T

Note: The problems in this HW set are the ones to point the direction we go from this course on. As such, you need a little ingenuity to solve them. Please wonder why these problems are given at the end of this course. You will find the answers as you advance in robotics.

[1] Solve the following problems:

(a) Show that if $x(t) = Ae^{-2t}$, then $\frac{dx(t)}{dt} + 2x(t) = 0$, where A is a constant.

(b) Show that if $x(t) = A\sin 3t$, then $\frac{d^2x(t)}{dt^2} + 9x(t) = 0$, where A is a constant.

(c) Find the value K if $x(t) = Ke^{-3t}$ and its derivatives satisfies the following equation: $\frac{d^2x(t)}{dt^2} + 4\frac{dx(t)}{dt} + 9x(t) = 24e^{-3t}$.

[2] A linear combination of a function and its derivatives is called an **<u>annihilator</u>** of a function if the linear combination becomes 0. An example is the one in Part (a) of the above problem, where $\frac{dx(t)}{dt} + 2x(t)$ is an annihilator for $x(t) = Ae^{-2t}$. Find an annihilator for the following functions.

(a) $x(t) = Be^{3t}$

(b) $x(t) = C\cos 2t$

(c) $x(t) = At + B$

(d) $x(t) = Ae^{-2t} + Be^{3t}$ (Hint: You need to annihilate 2 functions not 1)

[3] There exist non-zero vectors $\vec{x} \neq \vec{0}$ for any square matrix A such that the multiplication of A to $\vec{x}$, $A\vec{x}$, becomes a scalar multiple of $\vec{x}$, $k\vec{x}$, in other words, $A\vec{x} = k\vec{x}$.

(a) Show that $\vec{y} = \alpha\vec{x}$ also satisfies $A\vec{y} = k\vec{y}$ if $A\vec{x} = k\vec{x}$. (This means that the norm of $\vec{x}$ is not the matter, but the direction of $\vec{x}$ is.)

(b) Let us find a vector $\vec{x}$ and a scaler k for $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$. First, let $\vec{x} = [a\ \ b]^T$ and substitute it to the equation $A\vec{x} = k\vec{x}$. Then find k assuming that $a \neq 0 \neq b$. (Hint: there are 2 values for k.)

(c) Find the relationship between a and b for each k value in the form of $b = \cdots$.

(d) Substitute the expression you found in (c) to $\vec{x}$ and express $\vec{x}$ in the form $\vec{x} = a\vec{z}$ for each k value.

(e) Check if the vectors you found in (d) actually satisfy the equation $A\vec{x} = k\vec{x}$. Here, we can assume $a = 1$, WLOG.

(f) (Extra) It is a fact that both the scalar k and the vector $\vec{x}$ can be complex. This is because the real number system is not complete, but the complex number system is. See that this is the case using the following matrix.

$$A = \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}$$